

# Topology of complete Finsler manifolds admitting convex functions <sup>\*†</sup>

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## 1 Introduction.

Let  $(M, F)$  be an  $n$ -dimensional Finsler manifold. The well known Hopf-Rinow theorem (see for example [2]) states that  $M$  is complete if and only if the exponential map  $\exp_p$  at some point  $p \in M$  (and hence for every point on  $M$ ) is defined on the whole tangent space  $T_p M$  to  $M$  at that point. This is equivalent to state that  $(M, F)$  is geodesically complete with respect to forward geodesics at every point on  $M$ . Throughout this article we assume that  $(M, F)$  is *geodesically complete with respect to forward geodesics*.

A function  $\varphi : (M, F) \rightarrow \mathbb{R}$  is said to be *convex* if and only if along every geodesic (forward and backward)  $\gamma : [a, b] \rightarrow (M, F)$ , the restriction  $\varphi \circ \gamma : [a, b] \rightarrow \mathbb{R}$  is convex:

$$\varphi \circ \gamma((1 - \lambda)a + \lambda b) \leq (1 - \lambda)\varphi \circ \gamma(a) + \lambda\varphi \circ \gamma(b), \quad 0 \leq \lambda \leq 1. \quad (1.1)$$

If the inequality in the above relation is strict for all  $\gamma$  and for all  $\lambda \in (0, 1)$ , then  $\varphi$  is called *strictly convex*. If the second order difference quotient, namely the quantity  $\{\varphi \circ \gamma(h) - \varphi \circ \gamma(-h) - 2\varphi \circ \gamma(0)\}/h^2$  is bounded away from zero on every compact set on  $M$  along all  $\gamma$ , then  $\varphi$  is called *strongly convex*. In the case when  $\varphi$  is at least  $C^2$ , its convexity can be written in terms of the Finslerian Hessian of  $\varphi$ , but we do not need to do this in the present paper.

If  $\varphi \circ \gamma$  is a convex function of one variable, then the function  $\varphi \circ \bar{\gamma}$  is also convex, where  $\bar{\gamma}$  is the reverse curve of  $\gamma$ . For a general Finsler metric if  $\gamma$  is a geodesic it does not mean that the inverse curve  $\bar{\gamma}$  is a geodesic also, but  $\varphi \circ \bar{\gamma}$  is convex and so is  $\varphi \circ \gamma$  as well.

Every non-compact manifold admits a complete (Riemannian or Finslerian) metric and a non-trivial smooth function which is convex with respect to this metric (see [7]).

If a non-trivial convex function  $\varphi : (M, F) \rightarrow \mathbb{R}$  is constant on an open set, then  $\varphi$  assumes its minimum on this open set and the number of components of a *level set*  $M_a^a(\varphi) := \varphi^{-1}(\{a\})$ ,  $a \geq \inf_M \varphi$  is equal to that of the boundary components of the minimum set of  $\varphi$ . Here we denote  $\inf_M \varphi := \inf\{\varphi(x) : x \in M\}$ .

A convex function  $\varphi$  is said to be *locally non-constant* if it is not constant on any open set of  $M$ . From now on we always assume that a convex function is locally non-constant.

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The purpose of this article is to investigate the topology of complete Finsler manifolds admitting (locally non-constant) convex functions  $\varphi : (M, F) \rightarrow \mathbb{R}$ . Convex functions on complete Riemannian manifolds have been fully discussed in [7] and others. Although the distance function on  $(M, F)$  is not symmetric and the backward geodesics do not necessarily coincide with the forward geodesics, we prove that most of the Riemannian results obtained in [7] have the Finsler extensions, as stated below.

We first discuss the topology of the Finsler manifold  $(M, F)$  admitting a convex function  $\varphi$ .

**Theorem 1.1 (compare Theorem F, [7])** *Let  $\varphi : (M, F) \rightarrow \mathbb{R}$  be a convex function. Assume that all of the levels of  $\varphi$  are compact.*

*If  $\inf_M \varphi$  is not attained, then there exists a homeomorphism*

$$H : M_a^a(\varphi) \times (\inf_M \varphi, \infty) \rightarrow M,$$

*for an arbitrary fixed number  $a \in (\inf_M \varphi, \infty)$ , such that*

$$\varphi(H(y, t)) = t, \quad \forall y \in M_a^a(\varphi), \quad \forall t \in (\inf_M \varphi, \infty).$$

*Moreover, if  $\lambda := \inf_M \varphi$  is attained, then  $M$  is homeomorphic to the normal bundle over  $M_\lambda^\lambda(\varphi)$  in  $M$ .*

Next, we discuss the case where  $\varphi$  has a disconnected level.

**Theorem 1.2 (compare Theorem A, [7])** *Let  $\varphi : (M, F) \rightarrow \mathbb{R}$  be a convex function. If  $M_c^c(\varphi)$  is disconnected for some  $c \in \varphi(M)$ , we then have*

- (1)  *$\inf_M \varphi$  is attained.*
- (2) *If  $\lambda := \inf_M \varphi$ , then  $M_\lambda^\lambda(\varphi)$  is a totally geodesic smooth hypersurface which is totally convex without boundary.*
- (3) *The normal bundle of  $M_\lambda^\lambda(\varphi)$  in  $M$  is trivial.*
- (4) *If  $b > \lambda$ , then the boundary of the  $b$ -sublevel set  $M^b(\varphi) := \{x \in M \mid \varphi(x) \leq b\}$  has exactly two components.*

The diameter function  $\delta : \varphi(M) \rightarrow \mathbb{R}_+$  plays an important role in this article and it is defined as follows:

$$\delta(t) := \sup\{d(x, y) \mid x, y \in M_t^t(\varphi)\}. \quad (1.2)$$

It is known from [7] that the diameter function  $\delta$  of a complete Riemannian manifold admitting a convex function is monotone non-decreasing. However it is not certain if it is monotone on a Finsler manifold. In Theorem 1.1, we do not use the monotone property but only the local Lipschitz property of  $\delta$  which is proved in Proposition 3.3.

We finally discuss the number of ends of a Finsler manifold  $(M, F)$  admitting a convex function  $\varphi$ . As stated above, the diameter function  $\delta$ , defined on the image of the convex function  $\varphi$ , may not be monotone. It might occur that a convex function defined on a

Finsler manifold  $(M, F)$  may simultaneously admit both compact and non-compact levels. This fact makes difficult to discuss the number of ends of the manifold  $(M, F)$ . However, we shall discuss all the possible cases and prove:

**Theorem 1.3 (compare Theorems C, D and G, [7])** *Let  $\varphi : (M, F) \rightarrow \mathbb{R}$  be a convex function.*

A. *Assume that  $\varphi$  admits a disconnected level.*

(A1) *If all the level of  $\varphi$  are compact, then  $M$  has two ends.*

(A2) *If all the levels of  $\varphi$  are non-compact, then  $M$  has one end.*

(A3) *If both compact and non-compact levels of  $\varphi$  exist simultaneously, then  $M$  has at least three ends.*

B. *Assume that all the levels of  $\varphi$  are connected and compact.*

(B1) *If  $\inf_M \varphi$  is attained, then  $M$  has one end.*

(B2) *If  $\inf_M \varphi$  is not attained, then  $M$  has two ends.*

C. *If all the levels are connected and non-compact, then  $M$  has one end.*

D. *Assume that all the levels of  $\varphi$  are connected and that  $\varphi$  admits both compact and non-compact levels simultaneously. Then we have:*

(D1) *If  $\inf_M \varphi$  is not attained, then  $M$  has two ends.*

(D2) *If  $\inf_M \varphi$  is attained, then  $M$  has at least two ends.*

E. *Finally, if  $M$  has two ends, then all the levels of  $\varphi$  are compact.*

**Remark 1.4** The supplementary condition that all of the levels of  $\varphi$  are simultaneously compact or non-compact in the hypothesis of Theorem 1.1 is necessary because we have not proved that the diameter function  $\delta$  is monotone non-decreasing for a Finsler manifold. If this property of monotonicity would hold good, then this assumption can be removed.

We summarize the historical background of convex and related functions on manifolds, G-spaces and Alexandrov spaces. Locally non-constant convex functions, affine functions and peakless functions have been investigated on complete Riemannian manifolds and complete non-compact Busemann G-spaces and Alexandrov spaces in various ways. The topology of Riemannian manifolds  $(N, g)$  admitting locally nonconstant convex functions, have been investigated in [7], [1], [8], [9]. The topology of Busemann G-surfaces admitting convex functions has been investigated in [11] and in [13]. It should be noted that convex functions on complete Alexandrov surfaces are *not continuous*. The notion of peakless functions introduced by Busemann [4] is similar to quasiconvex functions and weaker than convex functions, and has been discussed in [5] and [12]. The topology of complete manifolds admitting locally geodesically (strictly) quasiconvex and uniformly locally convex filtrations have been investigated by Yamaguchi [22],[23] and [24]. The isometry groups of complete Riemannian manifolds  $(N, g)$  admitting strictly convex functions have been

discussed in [21] and others. A well known classical theorem due to Cartan states that every compact isometry group on an Hadamard manifold  $H$  has an fixed point. This follows from a simple fact that the distance function to every point on  $H$  is strictly convex. Peakless functions and totally geodesic filtration on complete manifolds have been discussed in [12], [5], [22], [23], [24] and others.

A convex function on  $(N, g)$  is said to be *affine* if and only if the equality in (1.1) holds for all  $\gamma$  and for all  $\lambda \in (0, 1)$ . The splitting theorem for Riemannian manifolds have been investigated in [10]. Alexandrov spaces admitting affine functions have been established in [10], [14] and [15]. An overview on the convexity of Riemannian manifolds can be found in [3].

The properties of isometry groups on Finsler manifolds admitting convex functions will be discussed separately. We refer the basic facts in Finsler and Riemannian geometry to [2], [16], [6], [18].

## 2 Fundamental facts

The fundamental facts on convex sets and convex functions on  $(M, F)$  are summarized as follows. Most of these are trivial in the Riemannian case, but we consider useful to formulate and prove them in the more general Finslerian setting.

Let  $(M, F)$  be a complete Finsler manifold. At each point  $p \in M$ , the indicatrix  $\Sigma_p \subset T_p M$  at  $p$  is defined as

$$\Sigma_p := \{u \in T_p M \mid F(p, u) = 1\}.$$

The *reversibility function*  $\lambda : (M, F) \rightarrow \mathbb{R}^+$  of  $(M, F)$  is given as

$$\lambda(p) := \sup \{F(p, -X) \mid X \in \Sigma_p\}.$$

Clearly,  $\lambda$  is continuous on  $M$  and

$$\lambda(p) = \max \left\{ \frac{F(p, -X)}{F(p, X)} \mid X \in T_p M \setminus \{0\} \right\}.$$

Let  $C \subset M$  be a compact set. There exists a constant  $\lambda(C) > 0$  depending on  $C$  such that if  $p \in C$  and if  $X \in \Sigma_p$ , then

$$\frac{1}{\lambda(C)} F(p, X) \leq F(p, -X) \leq \lambda(C) \cdot F(p, X).$$

In particular, if  $\sigma : [0, 1] \rightarrow C$  is a smooth curve, then the length  $L(\sigma) := \int_0^1 F(\sigma(t), \dot{\sigma}(t)) dt$  of  $\sigma$  satisfies

$$\frac{1}{\lambda(C)} L(\sigma) \leq L(\sigma^{-1}) \leq \lambda(C) \cdot L(\sigma).$$

Here we set  $\sigma^{-1}(t) := \sigma(1 - t)$ ,  $t \in [0, 1]$  the reverse curve of  $\sigma$ .

It is well known (see for example [2]) that the topology of  $(M, F)$  as an inner metric space is equivalent to that of  $M$  as a manifold. For a compact set  $C \subset M$ , the inner metric  $d_F$  of  $(M, F)$  induced from the Finslerian fundamental function has the property:

$$\frac{1}{\lambda(C)} d_F(p, q) \leq d_F(q, p) \leq \lambda(C) \cdot d_F(p, q), \quad \forall p, q \in C.$$

Let  $\text{inj} : (M, F) \rightarrow \mathbb{R}_+$  be the *injectivity radius function* of the exponential map. Namely,  $\text{inj}(p)$  for a point  $p \in M$  is the maximal radius of the ball centred at the origin of the tangent space  $T_p M$  at  $p$  on which  $\exp_p$  is injective.

A classical result due to J. H. C. Whitehead [20] states that there exists a *convexity radius function*  $r : (M, F) \rightarrow \mathbb{R}$  such that if  $B(p, r) := \{x \in M \mid d(p, x) < r\}$  is an  $r$ -ball centered at  $p$ , then  $B(q, r') \subset B(p, r)$  for every  $q \in B(p, r(p))$  and for every  $r' \in (0, r(p))$  is *strongly convex*. Namely, the distance function to  $p$  is strongly convex along every geodesic in  $B(p, r)$ ,  $r \in (0, r(p))$ , if its extension does not pass through  $p$ .

A closed set  $U \subset M$  is called *locally convex* if and only if  $U \cap B(p, r)$ , for every  $x \in U$  and for some  $r \in (0, r(p))$ , is convex. Pay attention to the fact that this definition has sense only for closed sets, since every open set is obviously locally convex.

A set  $V \subset M$  is called *totally convex* if and only if every geodesic joining two points on  $V$  is contained entirely in  $V$ . A closed hemi-sphere in the standard sphere  $\mathbb{S}^n$  is locally convex and an open hemi-sphere is strongly convex, while  $\mathbb{S}^n$  itself is the only one totally convex set on it.

The minimum set of a convex function on  $(M, F)$  is totally convex, if it exists.

**Proposition 2.1** (see [17] ; Theorem 4.6). *Let  $C \subset M$  be a compact set. Let  $\lambda(C)$  be the reversibility constant of the compact Finsler space  $(C, F)$ . If  $r(C)$  and  $\text{inj}_C$  are the convexity and injectivity radii of  $C$ , respectively, we then have:*

$$r(C) \leq \frac{\lambda}{1 + \lambda} \text{inj}_C. \quad (2.1)$$

**Proposition 2.2** A convex function  $\varphi : (M, F) \rightarrow \mathbb{R}$  defined as in (1.1) is locally Lipschitz.

*Proof.* Let  $C \subset M$  be an arbitrary fixed compact set and  $C_1 := \{x \in M \mid d(C, x) \leq 1\}$ . Here we set  $d(C, x) := \min\{d(y, x) \mid y \in C\}$ . For points  $x, y \in C_1$  we denote by  $\gamma_{xy} : [0, d(x, y)] \rightarrow M$ ,  $\gamma_{yx} : [0, d(y, x)] \rightarrow M$  minimizing geodesics with  $\gamma_{xy}(0) = x$ ,  $\gamma_{xy}(d(x, y)) = y$  and  $\gamma_{yx}(0) = y$ ,  $\gamma_{yx}(d(y, x)) = x$ . The slope inequalities along the convex functions:  $\varphi \circ \gamma_{xy}|_{[0, d(x, y)+1]}$  and  $\varphi \circ \gamma_{yx}|_{[0, d(y, x)+1]}$  imply that if  $\Lambda := \sup_{C_1} \varphi$  and  $\lambda := \inf_{C_1} \varphi$  (see Figure 1), then

$$\frac{\varphi(y) - \varphi(x)}{d(x, y)} \leq \Lambda - \lambda, \quad \frac{\varphi(x) - \varphi(y)}{d(y, x)} \leq \Lambda - \lambda.$$

There exists a constant  $L = L(C) > 0$  such that

$$\sup\left\{\frac{d(x, y)}{d(y, x)} \mid x, y \in C\right\} \leq L.$$

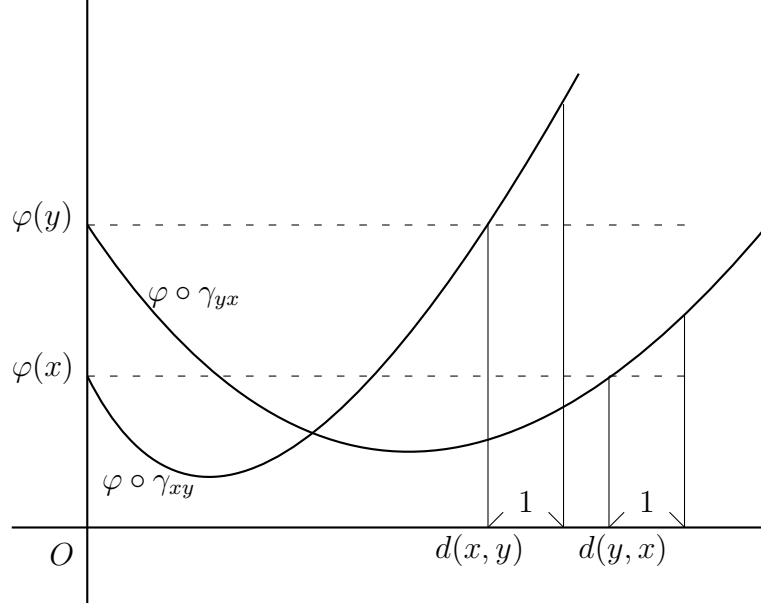


Figure 1: A convex function is locally Lipschitz.

Therefore we have

$$\left| \frac{\varphi(x) - \varphi(y)}{d(x, y)} \right|, \left| \frac{\varphi(y) - \varphi(x)}{d(y, x)} \right| \leq L(\Lambda - \lambda).$$

□

**Proposition 2.3** If  $C \subset (M, F)$  is a closed locally convex set, then there exists a  $k$ -dimensional totally geodesic submanifold  $W$  of  $M$  contained in  $C$  and its closure coincides with  $C$ .

*Proof.* Let  $r : (M, F) \rightarrow \mathbb{R}$  be the convexity radius function. For every point  $p \in C$  there exists a  $k(p)$ -dimensional smooth submanifold of  $M$  which is contained entirely in  $C$  and such that  $k(p)$  is the maximal dimension of all such submanifolds in  $C$ , where  $0 \leq k(p) \leq n$ . At least  $\{p\}$  is a 0-dimensional such a submanifold contained in  $C$ .

Let  $K \subset M$  be a large compact set containing  $p$  and  $r(K)$  the convexity radius of  $K$ , namely  $r(K) := \min\{r(x) \mid x \in K\}$ . Let  $k := \max\{k(p) \mid p \in C\}$ .

Let  $W(p) \subset C$  be a  $k$ -dimensional smooth submanifold of  $M$ . Suppose that  $W(p) \cap B(p; r) \subsetneq C \cap B(p; r)$  for a sufficiently small  $r \in (0, r(K))$ . Then, there exists a point  $q \in B(p; r) \cap (C \setminus W(p))$ . Clearly  $\dot{\gamma}_{pq}(0)$  is transversal to  $T_p W(p)$ , and hence a family of minimizing geodesics  $\{\gamma_{xq} : [0, d(x, q)] \rightarrow B(p; r) \mid x \in W(p) \cap B(p; r)\}$  with  $\gamma_{xq}(0) = x$ ,  $\gamma_{xq}(d(x, q)) = q$  has the property that every  $\dot{\gamma}_{xq}(0)$  is transversal to  $T_x W(p)$ . Therefore, this family of geodesics forms a  $(k + 1)$ -dimensional submanifold contained in  $C$ , a contradiction to the choice of  $k$ . This proves  $W(p) \cap B(p; r) = C \cap B(p; r)$  for a sufficiently small  $r \in (0, r(K))$ . We then observe that  $\cup_{p \in C} W(p) =: W \subset C$  forms a  $k$ -dimensional

smooth submanifold which is totally geodesic. Indeed, for any tangent vector  $v$  to  $W$ , there exists  $p \in C$  such that  $v \in T_p W(p)$  and due to the convexity of  $C$ , the geodesic  $\gamma_v : [0, \varepsilon] \rightarrow M$  cannot leave the submanifold  $W$ .

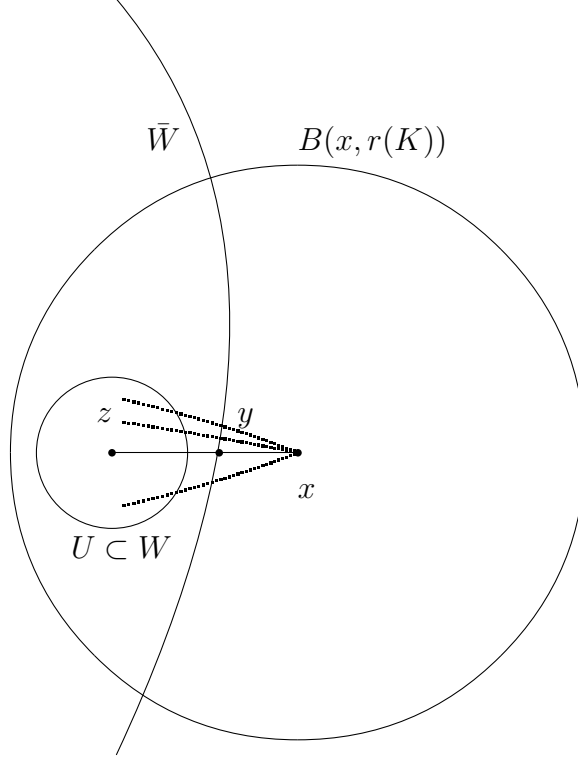


Figure 2: The closure  $\bar{W}$  of  $W$  coincides with  $C$ .

We finally prove that the closure  $\bar{W}$  of  $W$  coincides with  $C$ . Indeed, suppose that there exists a point  $x \in C \setminus \bar{W}$ . We then find a point  $y \in \bar{W} \setminus W$  such that  $d(x, y) = d(x, \bar{W}) < r(K)$ . If  $\dot{\gamma}_{xy}(d(x, y)) \in T_y \bar{W} := \lim_{y_j \rightarrow y} T_{y_j} M$ , then  $\gamma_{xy}(d(x, y) + \varepsilon) \in W$  for a sufficiently small  $\varepsilon > 0$ . Let  $U \subset W \cap B(x, r(K))$  be an open set around  $\gamma_{xy}(d(x, y) + \varepsilon)$ .

Then a family of geodesics

$$\{\gamma_{xz} : [0, d(x, z)] \rightarrow B(x, r(K)) \mid z \in U\} \quad (2.2)$$

forms a  $k$ -dimensional submanifold contained in  $W$  and hence  $y \in W$ , a contradiction to  $y \in \bar{W} \setminus W$ . Therefore,  $\dot{\gamma}_{xy}(d(x, y))$  does not belong to  $T_y \bar{W}$ , and (2.2) again forms a  $(k + 1)$ -dimensional submanifold in  $C$ , a contradiction to the choice of  $k$  (see Figure 2).  $\square$

Let  $C \subset M$  be a closed locally convex set and  $p \in C$ . There exists a totally geodesic submanifold  $W \subset C$  as stated in Proposition 2.3. We call  $W$  the *interior* of  $C$  and denote it by  $\text{Int}(C)$ . The *boundary* of  $C$  is defined by  $\partial C := C \setminus \text{Int}(C)$ , and the *dimension* of  $C$  is defined by  $\dim C := \dim \text{Int}(C)$ . The *tangent cone*  $\mathcal{C}_p(C) \subset T_p M$  of  $C$  at a point  $p \in C$  is defined as follows:

$$\mathcal{C}_p(C) := \{\xi \in T_p M \mid \exp_p t\xi \in \text{Int}(C), \text{ for some } t > 0\}. \quad (2.3)$$

Clearly,  $\mathcal{C}_p(C) = T_p \text{Int}(C) \setminus \{0\}$  for  $p \in \text{Int}(C)$ .

We also define the tangent space  $T_p C$  of  $C$  at a point  $p \in \partial C$  by  $T_p C := \lim_{q \rightarrow p} T_q \text{Int}(C)$ .

We claim that there exists for every point  $p \in \partial C$  an open half space  $T_p C_+ \subset T_p C$  of  $T_p C$  such that  $\mathcal{C}_p(C)$  is contained entirely in an open half space  $T_p C_+ \subset T_p C$  :

$$\mathcal{C}_p(C) \subset T_p C_+ \subset T_p C := \lim_{q \rightarrow p} T_q \text{Int}(C), \quad q \in \text{Int}(C). \quad (2.4)$$

Indeed, let  $p \in \partial C$  and  $\gamma_{qp} : [0, d(q, p)] \rightarrow B(p; r(K))$  for every point  $q \in B(p; r(K)) \cap \text{Int}(C)$  be a minimizing geodesic. Suppose that there is a point  $q \in \text{Int}(C)$  such that  $z := \gamma_{qp}(d(q, p) + \varepsilon) \in C$  for a sufficiently small  $\varepsilon > 0$ . We then have  $\dot{\gamma}_{qp}(d(q, p)) \in T_p C$ , and hence the convex cone as obtained in (2.2) is contained in  $C$ , a contradiction to the choice of  $p \in \partial C$ . From the above argument we observe that if  $p \in \partial C$ , then there exists a hyperplane  $H_p \subset T_p C$  such that  $\mathcal{C}_p(C)$  is contained in a half space  $T_p(C)_+ \subset T_p C$  bounded by  $H_p$ .

**Proposition 2.4** Let  $\varphi : (M, F) \rightarrow \mathbb{R}$  be a convex function. Then,  $M_a^a(\varphi)$  for every  $a > \inf_M \varphi$  is an embedded topological submanifold of dimension  $n - 1$ .

*Proof.* Let  $p \in M_a^a(\varphi)$  and  $q \in B(p; r(p)) \cap \text{Int}(M^a(\varphi))$ . There exists a hyperplane  $H_p \subset T_p M$  such that

$$H_p = \partial T_p(M^a(\varphi))_+ \quad \text{and} \quad \mathcal{C}_p(M^a(\varphi)) \subset T_p(M^a(\varphi))_+.$$

Every point  $x \in \exp_p(H_p) \cap B(p; r(p))$  is joined to  $q$  by a unique minimizing geodesic  $\gamma_{qx} : [0, d(q, x)] \rightarrow M$  such that  $\gamma_{qx}(0) = q$ ,  $\gamma_{qx}(d(q, x)) = x$ . Then there exists a unique parameter  $t(x) \in (0, d(q, x)]$  such that  $\gamma_{qx}(t(x)) \in M_a^a(\varphi) \cap B(p; r(p))$ . Let  $B_H(O; r(p))$  be the open  $r(p)$ -ball in  $H_p$  centered at the origin  $O$  of  $M_p$ . We then have a map  $\alpha_p : B_H(O; r(p)) \rightarrow M_a^a(\varphi)$  such that

$$\alpha_p(u) := \gamma_{qx}(t(x)), \quad u \in B_H(O; r(p)), \quad \exp_p u = x.$$

Clearly,  $\alpha_p$  gives a homeomorphism between  $B_H(O; r(p))$  and its image in  $M_a^a(\varphi)$ . Thus the family of maps  $\{(B_H(O; r(p)), \alpha_p) \mid p \in M_a^a(\varphi)\}$  forms an atlas of  $M_a^a(\varphi)$  (see Figure 3).

□

### 3 Level sets configuration

We shall give the proofs of Theorems 1.2 and 1.3. The following Lemma 3.1 is elementary and useful for our discussion.

**Lemma 3.1** Let  $\varphi : (M, F) \rightarrow \mathbb{R}$  be a convex function. If  $M_a^a(\varphi)$  is compact, then so is  $M_b^b(\varphi)$  for all  $b \geq a$ . If  $M_a^a(\varphi)$  is non-compact, then so is  $M_b^b(\varphi)$  for all  $b \leq a$ .



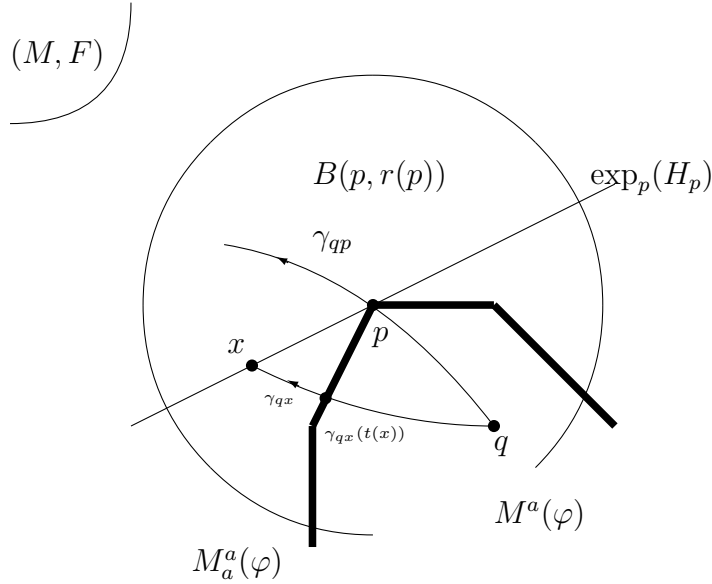


Figure 3: An atlas of local charts at an arbitrary point  $p \in M^a(\varphi)$ .

*Proof.* First of all we prove that if  $M^a(\varphi)$  is compact, then so is  $M^b(\varphi)$  for all  $b \geq a$ .

Suppose that  $M^b(\varphi)$  is noncompact for some  $b > a$ . Take a point  $p \in M^a(\varphi)$  and a divergent sequence  $\{q_j\}$ ,  $j = 1, 2, \dots$  on  $M^b(\varphi)$ . Since  $M^a(\varphi)$  is compact, there is a positive number  $L$  such that  $d(p, x) < L$  for all  $x \in M^a(\varphi)$ . Let  $\gamma_j : [0, d(p, q_j)] \rightarrow M$  be a minimizing geodesic with  $\gamma_j(0) = p$ ,  $\gamma_j(d(p, q_j)) = q_j$  for  $j = 1, 2, \dots$ . Compactness of  $M^a(\varphi)$  implies that each  $\varphi \circ \gamma_j|_{[L, d(p, q_j) - L]}$  is monotone and non-decreasing for all large numbers  $j$ .

Choosing a subsequence  $\{\gamma_i\}$  of  $\{\gamma_j\}$  if necessary, we find a ray  $\gamma_\infty : [0, \infty) \rightarrow M$  emanating from  $p$  such that  $\varphi \circ \gamma_\infty$  is monotone, non-decreasing and bounded above, and hence is constant  $= a$ . This contradicts the assumption that  $M^a(\varphi)$  is compact.  $\square$

The following Proposition 3.2 is the basic part of the proof of Theorem 1.1. Under the assumptions in Theorem 1.1, we divide  $M$  into countable compact sets such that  $M = \bigcup_{j=-\infty}^{\infty} \varphi^{-1}[t_{j-1}, t_j]$ , where  $\{t_j\}$  is monotone increasing and  $\lim_{j \rightarrow -\infty} t_j = \inf_M \varphi$  (if  $\inf_M \varphi$  is not attained) and  $\lim_{j \rightarrow \infty} t_j = \infty$ . In the application of Proposition 3.2 to each  $\varphi^{-1}[t_{j-1}, t_j]$ , the undefined numbers  $b_{k+1}$  and  $b_0$ , appearing in the proof of Proposition 3.2, play the role of margin to be pasted with  $\varphi^{-1}[t_j, t_{j+1}]$  (using  $b_{k+1}$ ) and with  $\varphi^{-1}[t_{j-2}, t_{j-1}]$  (using  $b_{-1}$ ), respectively.

**Proposition 3.2** *Let  $M^a(\varphi) \subset M$  be a connected and compact level set and  $b > a$  a fixed value. Then there exists a homeomorphism  $\Phi_a^b : M^b(\varphi) \times [a, b] \rightarrow M^b(\varphi)$  such that*

$$\varphi \circ \Phi_a^b(x, t) = t, \quad (x, t) \in M^b(\varphi) \times [a, b]. \quad (3.1)$$

*Proof.* Let  $K \subset M$  be a compact set with  $M^a(\varphi) \subset \text{Int}(K)$  and  $r := r(K)$  the convexity radius over  $K$ . We define two divisions as follows. Let  $a = a_0 < a_1 < \dots < a_k = b$  and  $b_{-1} < b_0 < \dots < b_k$  be given such that  $\varphi^{-1}[b_{-1}, b_k] \subset \text{Int}(K)$  and

1.  $b_{-1} < a_0 < b_0 < a_1 < \dots < a_{k-1} < b_{k-1} < a_k = b < b_k$ ,

$$2. \ b_j := \frac{a_j + a_{j+1}}{2}, \quad j = 0, 1, \dots, k-1,$$

$$3. \ \varphi^{-1}(\{a_{j-1}\}) \subset \bigcup \{B(x, r) \mid x \in \varphi^{-1}(\{a_{j+1}\})\}, \quad j = 1, \dots, k-1$$

$$4. \ \varphi^{-1}(\{b_{-1}\}) \subset \bigcup \{B(y, r) \mid y \in \varphi^{-1}(\{a_1\})\},$$

$$5. \ \varphi^{-1}(\{a_{k-1}\}) \subset \bigcup \{B(z, r) \mid z \in \varphi^{-1}(\{b_k\})\}.$$

Obviously we have  $[a, b] \subset [b_{-1}, b_k]$ .

For an arbitrary fixed point  $p'_j \in \varphi^{-1}(\{a_{j+1}\})$ , we have a minimizing geodesic  $T(p'_j, q_j)$  realizing the distance  $d(p'_j, \varphi^{-1}(-\infty, a_{j-1}])$  and  $q_j$  the foot of  $p'_j$  on  $\varphi^{-1}(-\infty, a_{j-1}]$ . Then the family of all such minimizing geodesics emanating from all the points on  $\varphi^{-1}(\{a_{j+1}\})$  to the points on  $\varphi^{-1}(\{a_{j-1}\})$  simply covers the set  $\varphi^{-1}[b_{j-1}, b_j]$ ,  $j = 1, 2, \dots, k$ . We define  $p_j := T(p'_j, q_j) \cap \varphi^{-1}(\{b_j\})$  and  $p_{j-1} := T(p'_j, q_j) \cap \varphi^{-1}(\{b_{j-1}\})$ . Once the point  $p_{j-1}$  has been defined, we then choose  $p'_{j-1} \in \varphi^{-1}(\{a_j\})$  and  $q_{j-1} \in \varphi^{-1}(\{a_{j-2}\})$  in such a way that  $T(p'_{j-1}, q_{j-1})$  realizes the distance  $d(p'_{j-1}, q_{j-1}) = d(p_{j-1}, \varphi^{-1}(-\infty, a_{j-2}])$  and it contains  $p_{j-1}$  in its interior. We thus obtain the inductive construction of a sequence  $\{T(p'_j, q_j) \mid j = 1, \dots, k\}$  of minimizing geodesics.

We finally choose a point  $p'_0 \in \varphi^{-1}(\{a_1\})$  and  $q_0 \in \varphi^{-1}(\{b_{-1}\})$  such that  $T(p'_0, q_0)$  is a unique minimizing geodesic with  $q_0$  being the foot of  $p'_0$  on  $\varphi^{-1}(-\infty, b_{-1}]$ . If we set  $q'_0 := \varphi^{-1}(\{a_0\}) \cap T(p'_0, q_0)$ , then  $d(p_0, q_1) \leq d(p_0, q'_0)$  follows from the fact that  $q_1$  is the foot of  $p_0$  on  $\varphi^{-1}(-\infty, a_0]$ . Therefore the slope inequality along  $T(p'_0, q_0)$  implies

$$\frac{a_0 - b_0}{d(p_0, q_1)} \leq \frac{a_0 - b_0}{d(p_0, q'_0)} \leq \frac{b_{-1} - a_0}{d(q'_0, q_0)},$$

and hence there exists a positive number

$$\Delta_a^b(K) := \min \left\{ \frac{a - b_{-1}}{d(q'_0, \varphi^{-1}(-\infty, b_{-1}])} \mid q'_0 \in \varphi^{-1}(\{a\}) \right\}$$

with the property that all the slopes of  $\varphi \circ T(p'_j, q_j)$  for every  $j = 0, 1, \dots, k$  are negative and bounded above by  $-\Delta_a^b(K)$ .

We define a broken geodesic  $T(p_k) := T(p_k, p_{k-1}) \cup \dots \cup T(p_1, p_0)$ ,  $p_k \in \varphi^{-1}(\{b_k\})$ , with its break points at  $p_j \in \varphi^{-1}(\{b_j\})$ ,  $j = 0, 1, \dots, k-1$  in such a way that each  $T(p_j, p_{j-1})$  is a proper subarc of a unique minimizing geodesic  $T(p'_j, q_j)$ , where  $q_j$  is the foot of  $p'_j$  on  $\varphi^{-1}(-\infty, a_{j-1}]$  (see Figure 4). Then  $T(p_{j-1}, p_{j-2})$  is a proper subarc of  $T(p'_{j-1}, q_{j-1})$ . Clearly, the convex function along  $T(p_k)$  is monotone strictly decreasing, since the slopes along  $\varphi \circ T(p_k)$  are all bounded above by  $-\Delta_a^b(K)$ . We then observe from the construction that the family of all the broken geodesics emanating from all points on  $\varphi^{-1}(\{b_k\})$  and ending at points on  $\varphi^{-1}(\{b_{-1}\})$  simply covers  $\varphi^{-1}[a, b]$ . The desired homeomorphism  $\Phi_a^b$  is now obtained by defining  $\Phi_a^b(x, t)$  as the intersection of a  $T(x)$  emanating from  $x : \Phi_a^b(x, t) = T(x) \cap \varphi^{-1}(\{t\})$ .  $\square$

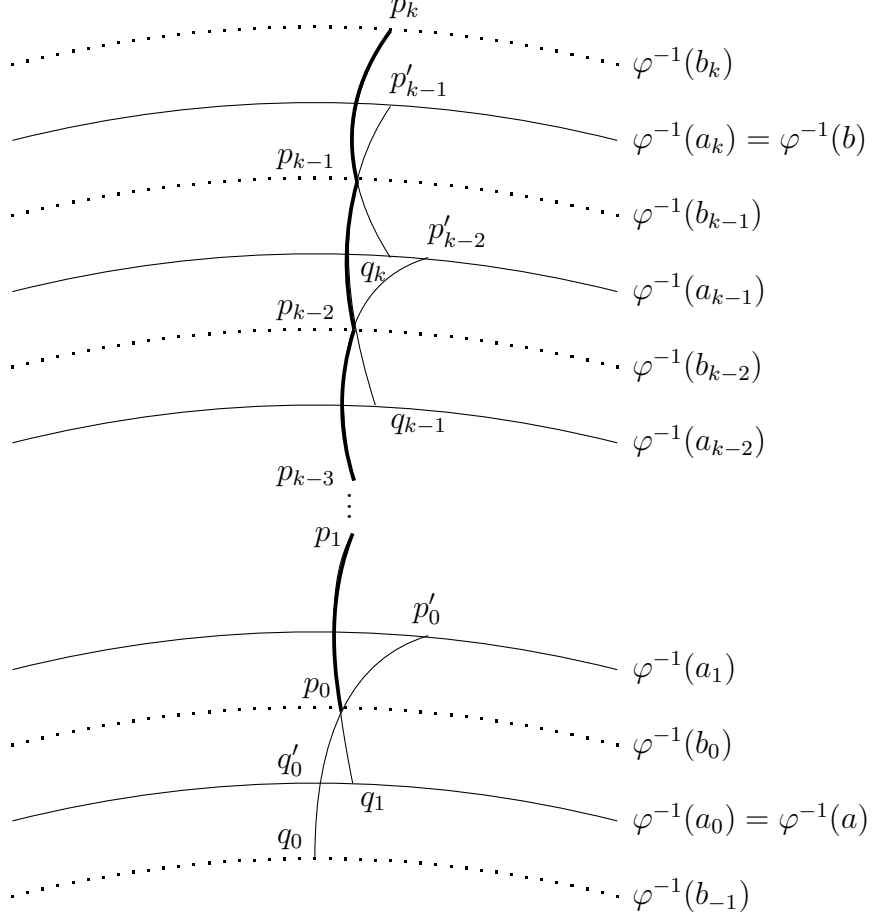


Figure 4: The broken geodesic  $T(p_k)$ .

**Proposition 3.3** Assume that all the levels of  $\varphi$  are compact. Then the diameter function  $\delta : \varphi(M) \rightarrow \mathbb{R}$  defined by

$$\delta(a) := \sup\{d(x, y) \mid x, y \in \varphi^{-1}(\{a\}), \quad a \in \varphi(M)\}$$

is locally Lipschitz.

*Proof.* Let  $\inf_M \varphi < a < b < \infty$  and  $r = r(M_a^b(\varphi))$  the convexity radius over  $M_a^b(\varphi)$ . Let  $x, y \in \varphi^{-1}(\{s\})$  for  $s \in [a, b]$  be such that  $d(x, y) = \delta(s)$ . Proposition 3.2 then implies that there are points  $x', y' \in M_b^b(\varphi)$  such that  $\Phi_a^b(x', s) = x$  and  $\Phi_a^b(y', s) = y$ . Moreover we have  $\Phi_a^b(x', t) = T(x') \cap M_t^t(\varphi)$  and  $\Phi_a^b(y', t) = T(y') \cap M_t^t(\varphi)$  and the length  $L(T(p)|_{[s, t]})$  of  $T(p)|_{[s, t]}$ , for  $p \in M_b^b(\varphi)$  and for every  $a \leq s < t \leq b$ , is bounded above by

$$L(T(p)|_{[s, t]}) \leq |t - s| / \Delta_a^b(M_a^b(\varphi)). \quad (3.2)$$

We therefore have by setting  $\lambda = \lambda(M_a^b(\varphi))$  the reversibility constant on  $M_a^b(\varphi)$ ,

$$\begin{aligned} \delta(s) &= d(x, y) \leq d(x, \Phi_a^b(x', t)) + d(\Phi_a^b(x', t), \varphi_a^b(y', t)) + d(\Phi_a^b(y', t), y) \\ &\leq \lambda|t - s| / \Delta_a^b(M_a^b(\varphi)) + \delta(t) + |t - s| / \Delta_a^b(M_a^b(\varphi)) \\ &= (1 + \lambda)|t - s| / \Delta_a^b(M_a^b(\varphi)) + \delta(t). \end{aligned}$$

Similarly, we obtain by choosing  $x, y \in \varphi^{-1}(\{t\})$ ,  $d(x, y) = \delta(t)$ ,

$$\begin{aligned}\delta(t) = d(x, y) &\leq d(x, \Phi_a^b(x', s)) + d(\Phi_a^b(x', s), \Phi_a^b(y', s)) + d(\Phi_a^b(y', s), y) \\ &\leq (1 + \lambda)|t - s|/\Delta_a^b(M_a^b(\varphi)) + \delta(s),\end{aligned}$$

and hence,

$$|\delta(t) - \delta(s)| \leq (1 + \lambda)|t - s|/\Delta_a^b(M_a^b(\varphi)).$$

□

*The proof of Theorem 1.1.*

We first assume that  $\inf_M \varphi$  is not attained. Let  $\{a_j\}_{j=0, \pm 1, \dots}$  be a monotone increasing sequence of real numbers with  $\lim_{j \rightarrow -\infty} a_j = \inf_M \varphi$  and  $\lim_{j \rightarrow \infty} a_j = \infty$ . We then apply Proposition 3.2 to each integer  $j$  and obtain a homeomorphism  $\Phi_j^{j+1} : \varphi^{-1}(\{a_{j+1}\}) \times (a_j, a_{j+1}] \rightarrow M_{a_j}^{a_{j+1}}$  such that

$$\varphi \circ \Phi_j^{j+1}(x, t) = t, \quad x \in \varphi^{-1}(\{a_{j+1}\}), \quad t \in (a_j, a_{j+1}]$$

The composition of these homeomorphisms gives the desired homeomorphisms  $\varphi : \varphi^{-1}(\{a\}) \times (\inf_M \varphi, \infty) \rightarrow M$ .

If  $\lambda := \inf_M \varphi$  is attained, then  $M_\lambda^\lambda(\varphi)$  is a  $k$ -dimensional totally geodesic submanifold which is totally convex and  $0 \leq k \leq \dim M - 1$ . A tubular neighborhood  $B(M_\lambda^\lambda(\varphi), r(M_\lambda^\lambda(\varphi)))$  around the minimum set is a normal bundle over  $M_\lambda^\lambda(\varphi)$  in  $M$  and its boundary  $\partial B(M_\lambda^\lambda(\varphi), r(M_\lambda^\lambda(\varphi)))$  is homeomorphic to a level of  $\varphi$ . Therefore  $M$  is homeomorphic to the normal bundle over the minimum set in  $M$ . This proves Theorem 1.1.

□

**Remark 3.4** Under the assumption in Theorem 1.1, it is not certain whether or not  $\lim_{t \rightarrow \inf_M \varphi} \delta(t) = \infty$ . It might happen that every level set above infimum is compact but the minimum set is non-compact. We do not know such an example on a Finsler manifold.

**Remark 3.5** The basic difference of treatments of convex functions between Riemannian and Finsler geometry can be interpreted as follows.

In the case where  $\varphi : (M, g) \rightarrow \mathbb{R}$  is a convex function with non-compact levels, the homeomorphism  $\Phi_a^b : M_b^b(\varphi) \times [a, b] \rightarrow M_a^b(\varphi)$  is obtained as follows. Fix a point  $p \in M_a^a(\varphi)$  and a sequence of  $R_j$ -balls centered at  $p : \{B(p, R_j) \mid j = 1, 2, \dots\}$  with  $\lim_{j \rightarrow \infty} R_j = \infty$ . Setting  $K_j$  for  $j = 1, 2, \dots$  the closure of  $B(p, R_j)$ , we find a sequence of constants  $\Delta_j := \Delta_a^b(K_j)$ . If  $x \in K_j \cap M_b^b(\varphi)$  is a fixed point, we then have a broken geodesic  $T(x) := T(x_k, x_{k-1}) \cup \dots \cup T(x_1, x_0)$  as obtained in the proof of Proposition 3.2, where  $x_0 \in M_a^a(\varphi)$ . The special properties of Riemannian distance function now applies to  $T(x_j, x_{j-1}) : [0, d(x_j, x_{j-1})] \rightarrow (M, g)$  to obtain that the distance function from  $p \in M_a^a(\varphi)$ ,  $t \mapsto d(p, T(x_j, x_{j-1}))(t)$  is strictly monotone decreasing. Here  $T(x_j, x_{j-1})$  is parameterized by arc-length such that  $T(x_j, x_{j-1})(0) = x_j$  and  $T(x_j, x_{j-1})(d(x_j, x_{j-1})) = x_{j-1}$ . Therefore we observe that  $T(x)$  is contained entirely in  $K_j$  and moreover the length  $L(T(x))$  of  $T(x)$  satisfies

$$L(T(x)) \leq (b - a)/\Delta_j, \quad \forall x \in K_j \cap M_b^b(\varphi).$$

If  $y_0 \in M_a^a(\varphi) \cap K_j$  is an arbitrary fixed point, Proposition 3.2 again implies that there exists a point  $y = y_m \in M_b^b(\varphi)$  such that  $T(y) = T(y_k, y_{k-1}) \cup \dots \cup T(y_1, y_0)$  has length at most  $(b - a)/\Delta_j$ . Therefore we have

$$d(p, y) < R_j + (b - a)/\Delta_j + 1.$$

We therefore observe that the correspondence between  $M_b^b(\varphi)$  and  $M_a^a(\varphi)$ ,  $x \mapsto x_0$  through  $T(x)$  is bijective, and the desired homeomorphism is constructed.

However in the Finslerian case where all the levels of a convex function  $\varphi : (M, F) \rightarrow \mathbb{R}$  are non-compact, the correspondence between  $M_b^b(\varphi)$  and  $M_a^a(\varphi)$ ,  $x \mapsto x_0$  through  $T(x) = T(x_k, x_{k-1}) \cup \dots \cup T(x_1, x_0)$  may not be obtained. In fact, the monotone decreasing property of  $t \mapsto d(p, T(x_j, x_{j-1}))$  might not hold for a Finsler metric. Therefore  $T(x)$  for a point  $x \in K_j \cap M_b^b(\varphi)$  may not necessarily be contained in  $K_j$ , and hence, we may fail in controlling the length of  $T(x)$  in terms of  $\Delta_j$ . By the same reason, we cannot prove the monotone non-decreasing property of the diameter function for compact levels of a convex function  $\varphi : (M, F) \rightarrow \mathbb{R}$ .

## 4 Proof of Theorem 1.2

We take a minimizing geodesic  $\sigma : [0, \ell] \rightarrow M$  such that  $\sigma(0)$  and  $\sigma(\ell)$  belong to distinct components of  $M_c^c(\varphi)$ .

For the proof of (1), we assert that  $\inf_M \varphi = \inf_{0 \leq t \leq \ell} \varphi \circ \sigma(t)$ . Suppose  $b := \inf_{0 \leq t \leq \ell} \varphi \circ \sigma(t) > \inf_M \varphi$ . Since  $\varphi$  is locally non-constant, we may assume without loss of generality that  $b := \inf_{0 \leq t \leq \ell} \varphi \circ \sigma(t)$  is attained at a unique point, say,  $q = \sigma(\ell_0)$ .

Setting  $r = r(\sigma(\ell_0))$ , we find a number  $a \in (\inf_M \varphi, b)$  such that there is a unique foot  $p \in M_a^a(\varphi)$  of  $q$  on  $M_a^a(\varphi)$ , namely  $d(\sigma(\ell_0), M_a^a(\varphi)) = d(\sigma(\ell_0), p)$ .

Let  $\alpha : [0, d(q, p)] \rightarrow M$  be a unique minimizing geodesic with  $\alpha(0) = q$ ,  $\alpha(d(q, p)) = p$ . The points on  $\alpha(t)$ ,  $0 \leq t \leq d(q, p)$  can be joined to  $q_{\pm} := \sigma(\ell_0 \pm r)$  by a unique minimizing geodesic  $\gamma_{\alpha(t)q_{\pm}} : [0, d(\alpha(t), q_{\pm})] \rightarrow B(q; r)$  with  $\gamma_{\alpha(t)q_{\pm}}(0) = \alpha(t)$ ,  $\gamma_{\alpha(t)q_{\pm}}(d(\alpha(t), q_{\pm})) = q_{\pm}$ .

Since  $\varphi(q_{\pm}) > b$ , the right hand derivative of  $\varphi \circ \gamma_{\alpha(t)q_{\pm}}$  at  $d(\alpha(t), q_{\pm})$  is bounded below by

$$(\varphi \circ \gamma_{\alpha(t)q_{\pm}})'_+(\varphi \circ \gamma_{\alpha(t)q_{\pm}}(d(\alpha(t), q_{\pm}))) > \frac{\varphi(q_{\pm}) - b}{2r} > 0.$$

Thus, for every  $t \in [0, d(q, p)]$ ,  $\gamma_{\alpha(t)q_{\pm}}$  meets  $M_c^c(\varphi)$  at  $\gamma_{\alpha(t)q_{\pm}}(u^{\pm}(t))$  with

$$u^{\pm}(t) \leq \frac{2r(c - a)}{\varphi(q_{\pm}) - b} + 2r,$$

and hence there are curves  $C_0^{\pm} : [0, d(q, p)] \rightarrow M_c^c(\varphi)$  with  $C_0^+(0) = \sigma(\ell)$ ,  $C_0^-(0) = \sigma(0)$  and  $C_0^+(d(q, p)) = \gamma_{pq_+}(u^+(d(q, p)))$ ,  $C_0^-(d(q, p)) = \gamma_{pq_-}(u^-(d(q, p)))$ . Let  $\tau_t : [0, d(p, \sigma(t))] \rightarrow M$  for  $t \in [\ell_0 - r, \ell_0 + r]$  be a minimizing geodesic with  $\tau_t(0) = p$ ,

$\tau_t(d(p, \sigma(t))) = \sigma(t)$ . Every  $\tau_t$  meets  $M_c^c(\varphi)$  at a parameter value  $\leq (2rc)/(b-a)$ , and hence we have a curve  $C_1 : [\ell_0 - r, \ell_0 + r] \rightarrow M_c^c(\varphi)$  such that

$$C_1(t) = \tau_t[0, \frac{2rc}{b-a}] \cap M_c^c(\varphi).$$

Thus, considering the union  $C_0^- \cup C_1 \cup (C_0^+)^{-1}$ , it follows that  $\sigma(0)$  can be joined to  $\sigma(\ell)$  in  $M_c^c(\varphi)$ , a contradiction. This proves (1) (see Figure 5).

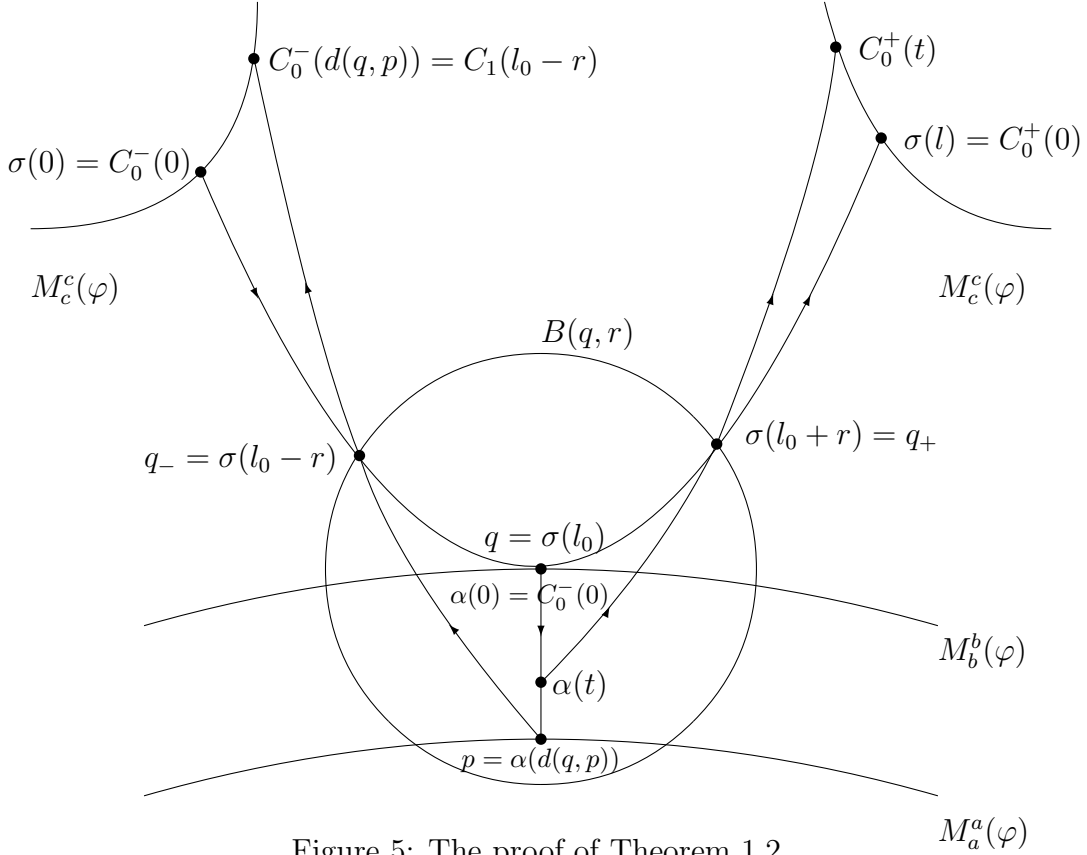


Figure 5: The proof of Theorem 1.2.

We next prove (2). Let  $\lambda := \inf_M \varphi$ . Clearly  $M_\lambda^\lambda(\varphi)$  is totally convex, and hence Proposition 2.3 implies that  $M_\lambda^\lambda(\varphi)$  carries the structure of a smooth totally geodesic submanifold.

Suppose that  $\dim M_\lambda^\lambda(\varphi) < n - 1$ , then the normal bundle is connected, and at each point  $p \in M_\lambda^\lambda(\varphi)$  the indicatrix  $\Sigma_p \subset T_p M$  has the property that  $\Sigma_p \setminus \Sigma_p(M_\lambda^\lambda(\varphi))$  is arcwise connected. Here,  $\Sigma_p(M_\lambda^\lambda(\varphi)) \subset \Sigma_p$  is the indicatrix at  $p$  of  $M_\lambda^\lambda(\varphi)$ . Choose points  $q_0$  and  $q_1$  on distinct components of  $M_c^c(\varphi)$ , and an interior point  $p \in M_\lambda^\lambda(\varphi)$ . If  $\gamma_i : [0, d(p, q_i)] \rightarrow M$  for  $i = 0, 1$  is a minimizing geodesic with  $\gamma_i(0) = p$ ,  $\gamma_i(d(p, q_i)) = q_i$ , then  $\dot{\gamma}_0(0)$  and  $\dot{\gamma}_1(0)$  are joined by a curve  $\Gamma : [0, 1] \rightarrow \Sigma_p \setminus \Sigma_p(M_\lambda^\lambda(\varphi))$  such that  $\Gamma(0) = \dot{\gamma}_0(0)$ ,  $\Gamma(1) = \dot{\gamma}_1(0)$ . The same method as developed in the proof of (1) yields a continuous 1-parameter family of geodesics  $\gamma_t : [0, \ell_t] \rightarrow M$  with  $\gamma_t(0) = p$ ,  $\dot{\gamma}_t(0) = \Gamma(t)$  and  $\gamma_t(\ell_t) \in M_c^c(\varphi)$  for all

$t \in [0, 1]$ . Thus we have a curve  $t \mapsto \gamma_t(\ell_t)$  in  $M_c^c(\varphi)$  joining  $q_0$  to  $q_1$ , a contradiction. This proves  $\dim M_\lambda^\lambda(\varphi) = n - 1$ .

We use the same idea for the proof of  $M_\lambda^\lambda(\varphi)$  having no boundary. In fact, the tangent cone of  $M_\lambda^\lambda(\varphi)$  at a boundary point  $x$  (supposing that the boundary is non-empty) is contained entirely in a closed half space of  $T_x M_\lambda^\lambda(\varphi)$ , and hence  $\Sigma_x \setminus \Sigma_x(M_\lambda^\lambda(\varphi))$  is arcwise connected. A contradiction is derived by constructing a curve in  $M_c^c(\varphi)$  joining  $q_0$  to  $q_1$ . This proves (2).

The triviality of the normal bundle over  $M_\lambda^\lambda(\varphi)$  in  $M$  is now clear.

We finally prove (4). Suppose that  $M_a^a(\varphi)$  for some  $a \in \varphi(M)$  has at least three components. Let  $q_1, q_2, q_3 \in M_a^a(\varphi)$  be taken from distinct components, and  $p \in M_\lambda^\lambda(\varphi)$ . Let  $\gamma_i : [0, d(p, q_i)] \rightarrow M$  for  $i = 1, 2, 3$  be minimizing geodesics with  $\gamma_i(0) = p$ ,  $\gamma_i(d(p, q_i)) = q_i$ . As is shown in (3), since the normal bundle over  $M_\lambda^\lambda(\varphi)$  in  $M$  is trivial, it follows that  $\Sigma_p \setminus \Sigma_p(M_\lambda^\lambda(\varphi))$  has exactly two components. Two of the three initial vectors, say  $\dot{\gamma}_1(0)$  and  $\dot{\gamma}_2(0)$  belong to the same component of  $\Sigma_p \setminus \Sigma_p(M_\lambda^\lambda(\varphi))$ . Then the same technique as developed in the proof of  $\dim M_\lambda^\lambda(\varphi) = n - 1$  applies, and  $q_1$  is joined to  $q_2$  by a curve in  $M_a^a(\varphi)$ , that is a contradiction. This proves (4).  $\square$

## 5 Ends of $(M, F)$

An *end*  $\varepsilon$  of a noncompact manifold  $X$  is an assignment to each compact set  $K \subset X$  a component  $\varepsilon(K)$  of  $X \setminus K$  such that  $\varepsilon(K_1) \supset \varepsilon(K_2)$  if  $K_1 \subset K_2$ . Every non-compact manifold has at least one end. For instance,  $\mathbb{R}^n$  has one end if  $n > 1$  and two ends if  $n = 1$ .

In the present section we discuss the number of ends of  $(M, F)$  admitting a convex function, namely we will prove Theorem 1.3. As is seen in the previous section, it may happen that a convex function  $\varphi : (M, F) \rightarrow \mathbb{R}$  has both compact and non-compact levels simultaneously. In this section let  $\{K_j\}_{j=1,2,\dots}$  be an increasing sequence of compact sets such that  $\lim_{j \rightarrow \infty} K_j = M$ .

We prove Theorem 1.3-(A1).

Theorem 1.2 (1) then implies that  $\varphi$  attains its infimum  $\lambda := \inf_M \varphi$ . For an arbitrary given compact set  $A \subset M$ , there exists a number  $a \in \varphi(M)$  such that  $M_a^a(\varphi)$  has two components and  $A \subset \varphi^{-1}[\lambda, a]$ . Then  $M \setminus A$  contains two unbounded open sets  $\varphi^{-1}(a, \infty)$ , proving (A-1).

We prove Theorem 1.3-(A2). Suppose that  $M$  has more than one end. There is a compact set  $K \subset M$  such that  $M \setminus K$  has at least two unbounded components, say,  $U$  and  $V$ . Setting  $a := \min_K \varphi$  and  $b := \max_K \varphi$ , we have

$$\lambda \leq a < b < \infty.$$

We assert that

$$M_\lambda^\lambda(\varphi) \cap U \neq \emptyset, \quad M_\lambda^\lambda(\varphi) \cap V \neq \emptyset, \quad M_\lambda^\lambda(\varphi) \cap K \neq \emptyset.$$

In order to prove that  $M_\lambda^\lambda(\varphi) \cap K \neq \emptyset$ , we suppose that  $\lambda < a$ .

Suppose the contrary, namely  $M_\lambda^\lambda(\varphi) \cap K = \emptyset$ . Once  $M_\lambda^\lambda(\varphi) \cap K \neq \emptyset$  has been established, it will turn out that  $M_\lambda^\lambda(\varphi)$  intersects all the unbounded components of  $M \setminus K$ .

Without loss of generality we may suppose that  $M_\lambda^\lambda(\varphi) \subset U$ . From Theorem 1.2 (3) it follows that  $M \setminus M_\lambda^\lambda(\varphi) = M_- \cup M_+$  (disjoint union with  $\partial M_+ = \partial M_- = M_\lambda^\lambda(\varphi)$ ).

Setting  $M_- \subset U$ , we observe that  $K \cup V \subset M_+$ .

If  $b_1 > b$ , then  $M_-$  contains a component of  $M_{b_1}^{b_1}(\varphi)$  and another component of  $M_{b_1}^{b_1}(\varphi)$  is contained entirely in  $V$ . We then observe that if  $\sup_{U \setminus M_-} \varphi = \infty$ , then  $U \setminus M_-$  contains a component of  $M_{b_1}^{b_1}(\varphi)$ , for  $\varphi$  takes value  $\leq b$  on  $\partial(U \setminus M_-)$  and  $M_{b_1}^{b_1}(\varphi)$  does not meet the boundary of  $U \setminus M_-$ . This contradicts to the Theorem 1.2 (4), for  $\partial M_{b_1}^{b_1}(\varphi)$  has at least three components. Therefore we have  $\sup_{U \setminus M_-} \varphi < \infty$ .

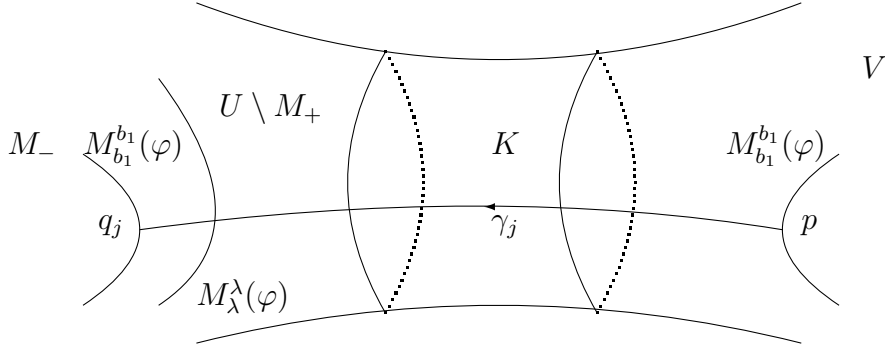


Figure 6: The proof of Theorem 1.3-(A2).

Let  $\{q_j\} \subset M_\lambda^\lambda(\varphi)$  be a divergent sequence of points and let us fix a point  $p \in M_{b_1}^{b_1}(\varphi) \subset V$ . Let  $\gamma_j : [0, d(p, q_j)] \rightarrow M \setminus M_-$  be a minimizing geodesic with  $\gamma_j(0) = p$ ,  $\gamma_j(d(p, q_j)) = q_j$ ,  $j = 1, 2, \dots$ . Clearly  $\gamma_j$  passes through a point on  $K$  and  $\varphi \circ \gamma_j$  is bounded above by  $b_1$ . If  $\gamma : [0, \infty) \rightarrow M \setminus M_-$  is a ray with  $\dot{\gamma}(0) = \lim_{j \rightarrow \infty} \dot{\gamma}_j(0)$ , then  $\varphi \circ \gamma$  is constant on  $[0, \infty)$  and  $\varphi \circ \gamma(t) = b_1$  for all  $t > b_1$ . This is a contradiction to the choice of  $b = \max_K \varphi$ , for  $\gamma$  passes through a point on  $K$  at which  $\varphi$  takes the value  $b_1$ . This proves the assertion (see Figure 6).

We next assert that if  $b_1 > b$  is fixed, then  $M_{b_1}^{b_1}(\varphi)$  has at least four components. In fact, we observe from  $M_{b_1}^{b_1}(\varphi) \cap K = \emptyset$ , each unbounded component of  $(M \setminus M_\lambda^\lambda(\varphi)) \cap (M \setminus K)$  contains a component of  $M_{b_1}^{b_1}(\varphi)$ . This contradicts Theorem 1.2 (4), and (A-2) is proved.

The proof of (A3) is a consequence of (D2), and given after the proof of (D2).

We prove Theorem 1.3-(B1). From assumption of (B1), it follows that  $\varphi^{-1}[\inf_M \varphi, b_j]$  is compact for all  $j$ , where  $\{b_j\}$  is a monotone divergent sequence. Then  $K_j := \varphi^{-1}[\lambda, b_j]$  is monotone increasing and  $\lim_{j \rightarrow \infty} K_j = M$ . Clearly  $M \setminus K_j$ , for every  $j = 1, 2, \dots$ , contains a unique unbounded domain  $\varphi^{-1}(b_j, \infty)$ . This proves Theorem 1.3-(B1).

We prove Theorem 1.3-(B2). From assumption of (B2), we have monotone sequences  $\{a_j\}$  and  $\{b_j\}$  such that

$$\lim_{j \rightarrow \infty} a_j = \inf_M \varphi, \quad \lim_{j \rightarrow \infty} b_j = \infty, \quad [a_j, b_j] = \varphi(K_j), \quad j = 1, 2, \dots$$



Then  $M \setminus K_j$  for all large number  $j$  contains two unbounded domains

$$M \setminus K_j \supset \varphi^{-1}(b_j, \infty) \cup \varphi^{-1}(\inf_M \varphi, a_j).$$

This proves that  $M$  has exactly two ends.

We prove Theorem 1.3-(C). We first prove (C) under an additional assumption that  $\lambda := \inf_M \varphi$  is attained. Suppose that  $M$  has more than one end. Using the same notation as in the proof of (A2),

$$\lambda := \inf_M \varphi, \quad a := \min_K \varphi, \quad b := \max_K \varphi,$$

where  $K \subset M$  is a compact set such that  $M \setminus K$  has at least two unbounded components  $U$  and  $V$ .

We first assert that  $K \cap M_\lambda^\lambda(\varphi) \neq \emptyset$ . In fact, supposing that  $K \cap M_\lambda^\lambda(\varphi) = \emptyset$  we find a component  $V$  of  $M \setminus K$  such that if  $b' > b$  then  $M_{b'}^{b'}(\varphi) \subset V$  and  $M_\lambda^\lambda \subset U$ . Here the assumption that all the levels of  $\varphi$  are connected is essential. As is seen in the proof of (A2), there exist at least two components of  $M_{b'}^{b'}$  for  $b' > b$  such that one component of it lies in  $U$  and another in  $V$ . This contradicts to the assumption in (C), and the first assertion is done.

The same proof technique as developed in (A2) implies that  $M_\lambda^\lambda(\varphi)$  passes through points on  $K$ ,  $U$  and  $V$ . Fix a point  $p \in V \cap M_\lambda^\lambda(\varphi)$  and a divergent sequence  $\{q_j\}$  of points in  $U \setminus M_\lambda^\lambda(\varphi)$  and  $\gamma_j : [0, d(p, q_j)] \rightarrow M$  a minimizing geodesic with  $\gamma_j(0) = p$ ,  $\gamma_j(d(p, q_j)) = q_j$ . From construction of  $\gamma_j$ , we observe that  $\varphi \circ \gamma_j$  is strictly increasing, and hence we find a number  $t_j > 0$  such that  $\gamma_j(t_j) \in M_{b'}^{b'}(\varphi) \cap U$ .

From the construction of  $\gamma_j$ , we observe that  $\varphi \circ \gamma$  is strictly increasing, and hence we find a number  $t_j > 0$  such that  $\gamma_j(t_j) \in M_{b'}^{b'}(\varphi) \cap U$ .

More precisely,  $\gamma_j[0, d(p, q_j)]$  meets  $M_\lambda^\lambda(\varphi)$  only at the origin, for  $M_\lambda^\lambda(\varphi)$  is totally convex and hence if  $\gamma(t_0) \in M_\lambda^\lambda(\varphi)$ , for some  $t_0 \in [0, d(p, q_j)]$ , then  $\gamma_j[0, d(p, q_j)]$  is contained entirely in  $M_\lambda^\lambda(\varphi)$ .

Therefore  $M_{b'}^{b'}(\varphi)$  has more than one components (one component in  $U$  and another component in  $V$ ), a contradiction to the assumption in (C). This concludes the proof of (C) in this case.

We next prove (C) in the case where  $\inf_M \varphi$  is not attained. Assume again that  $M$  has more than one end. We then have

$$\inf_M \varphi < a < b < \infty, \quad a := \min_K \varphi, \quad b := \max_K \varphi.$$

Since all the levels are connected, we find  $\inf_M \varphi < a' < a$  and  $b < b'$  such that  $M_{a'}^{a'}(\varphi) \subset U$  and  $M_{b'}^{b'}(\varphi) \subset V$ . Let  $\{y_j\} \subset M_{b'}^{b'}(\varphi)$  be a divergent sequence of points and fix a point  $x \in M_{a'}^{a'}(\varphi)$ . Let  $\gamma_j : [0, d(x, y_j)] \rightarrow M$  for  $j = 1, 2, \dots$  be a minimizing geodesic with  $\gamma_j(0) = x$  and  $\gamma_j(d(x, y_j)) = y_j$ . There exists a ray  $\gamma : [0, \infty) \rightarrow M$  emanating from  $x$  such that  $\dot{\gamma}(0) = \lim_{j \rightarrow \infty} \dot{\gamma}_j(0)$ . Clearly, every  $\gamma_j$  passes through a point on  $K$  and hence, so does  $\gamma$ . From construction,  $\varphi \circ \gamma : [0, \infty) \rightarrow \mathbb{R}$  is bounded from above by  $b'$ , and hence it is constant. However it is impossible, for  $\varphi(x) = a'$  and  $\varphi \circ \gamma(t_0) \geq a > a'$  at a point  $\gamma(t_0) \in K$ . This proves (C) in this case.

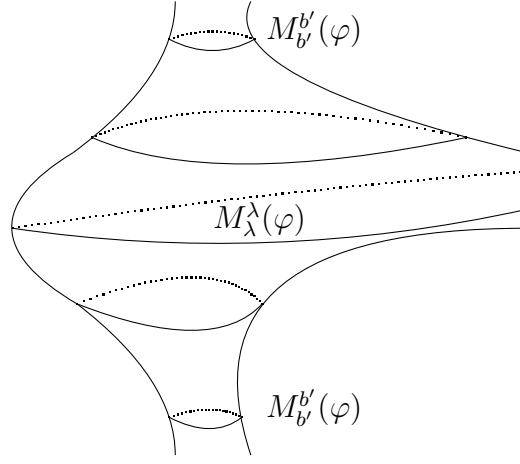


Figure 7: The proof of Theorem 1.3-(D1).

We prove Theorem 1.3-(D).

For the proof of (D1), we suppose that  $M$  has more than two ends.

Let  $K \subset M$  be a connected compact subset such that  $M \setminus K$  contains at least three unbounded components, say  $U$ ,  $V$  and  $W$ . We may consider that  $U$  contains  $\varphi^{-1}[b', \infty)$ , for all  $b' > b$ . Since all the levels of  $\varphi$  are connected, we have

$$\sup_{M \setminus U} \varphi \leq b.$$

In fact, suppose that there exists a point  $x \in M \setminus U$  such that  $\varphi(x) = b'$ , for some  $b' > b$ . Then  $M_{b'}^{b'}(\varphi) \cap K = \emptyset$  and hence  $M_{b'}^{b'}(\varphi)$  is disconnected, a contradiction to the assumption of (D).

Let  $\{x_j\} \subset V$  and  $\{y_j\} \subset W$  be two divergent sequences of points and  $\gamma_j : [0, d(x_j, y_j)] \rightarrow M \setminus U$  a minimizing geodesic joining  $x_j$  to  $y_j$ . Since  $\gamma_j$  passes through a point on  $K$ , there exists a straight line  $\gamma : \mathbb{R} \rightarrow M \setminus U$  such that  $\dot{\gamma}(0)$  is obtained as the limit of a converging sequence of vectors  $\dot{\gamma}_j(t_j) \in K$ , for  $j = 1, 2, \dots$ . Clearly,  $\varphi \circ \gamma : \mathbb{R} \rightarrow \mathbb{R}$  is bounded above, and hence constant taking a value  $\mu = \varphi \circ \gamma(0) \in [a, b]$ . We therefore observe that

$$M_{\mu}^{\mu}(\varphi) \cap K \neq \emptyset, \quad M_{\mu}^{\mu}(\varphi) \cap W \neq \emptyset \quad \text{and} \quad M_{\mu}^{\mu}(\varphi) \cap V \neq \emptyset.$$

We next choose a value  $a' \in (\inf_M \varphi, a)$ . We may assume without loss of generality that  $M_{a'}^{a'}(\varphi) \subset V$ . Let  $\{z_j\} \subset M_{\mu}^{\mu}(\varphi) \cap W$  be a divergent sequence of points and  $x \in M_{a'}^{a'}(\varphi)$  an arbitrary fixed point. Let  $\sigma_j : [0, d(x, z_j)] \rightarrow M \setminus U$  be a minimizing geodesic with  $\sigma_j(0) = x$ ,  $\sigma(d(x, z_j)) = z_j$ , for all  $j = 1, 2, \dots$ . Clearly,  $\varphi \circ \sigma_j$  is monotone increasing in  $W$ . Let  $\sigma : [0, \infty) \rightarrow M$  be a ray such that  $\dot{\sigma}(0) = \lim_{j \rightarrow \infty} \dot{\sigma}_j(0)$ . We then observe that  $\varphi \circ \sigma$  is monotone increasing on an unbounded interval  $[\bar{b}, \infty)$  for some  $\bar{b} > 0$ , and bounded above by  $\mu$ , and hence it is constant equal to  $a'$ . Recall that  $\varphi \circ \sigma(0) = \varphi(x) = a'$ . However this is impossible since  $a' < \min_K \varphi = a$  and  $\sigma[0, \infty)$  passes through a point on  $K$ . We therefore observe that  $M \setminus (K \cup U)$  has exactly one end. This proves (D1).

The proof of (D2) is now clear and omitted here.

The proof of (A3) is now a straightforward consequence of (D2), see Figure 8.

If  $M_b^b(\varphi)$  is compact for some  $b \in \varphi(M)$ , then  $\varphi^{-1}[b, \infty)$  has two ends. From the assumption and Theorem 1.2 (1), we observe that  $M_{\lambda}^{\lambda}(\varphi)$  is non-compact. Therefore  $M^b(\varphi) = \varphi^{-1}[\lambda, b]$  is non-compact and hence contains at least one end. This proves (A3).

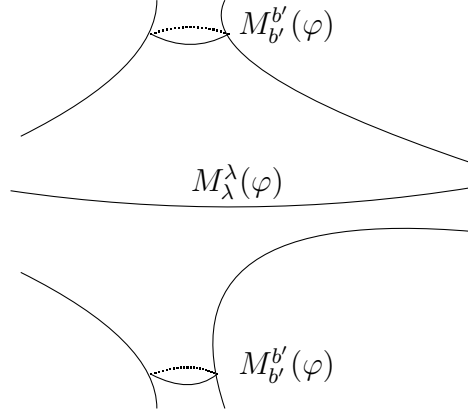


Figure 8: The proof of Theorem 1.3-(A3).

We prove Theorem 1.3 (E). Suppose that  $\varphi$  admits both compact and non-compact levels simultaneously. The same notation as in the proof of (D) will be used. If  $\varphi$  admits a disconnected level, then  $\varphi^{-1}[b', \infty)$ , for all  $b' > b$ , consists of two unbounded components.

Then Theorem 1.2 (1) and Lemma 3.1 imply that  $\lambda := \inf_M \varphi$  is attained and  $M_\lambda^\lambda(\varphi)$  is connected and noncompact. Therefore, every compact set  $K$  containing  $M_{b'}^{b'}(\varphi)$  has the property that  $M \setminus K$  has more than two unbounded components. In fact, two components of  $M \setminus K$  contain  $\varphi^{-1}[b', \infty)$  and the other component intersects with  $M_\lambda^\lambda(\varphi)$  outside  $K$ . This proves that  $M$  has at least three ends, a contradiction to the assumption of (E).

If all the levels of  $\varphi$  are connected and non-compact, then  $M$  has one end by (C), a contradiction to the assumption of (E). This completes the proof of (E).

## References

- [1] V. Bangert, *Riemannsche Mannigfaltigkeiten mit nicht-konstanten konvexer Funktionen*, Arch. Math., **31**, 163–170, (1978).
- [2] D. Bao–S. S. Chern–Z. Shen, *An Introduction to Riemannian-Finsler Geometry*, **200**, Graduate Texts in Math. Springer-Verlag, New York, 2000.
- [3] Yu. D. Burago–V. A. Zalgaller, *Convex sets in Riemannian Spaces of non-negative curvature*, Russian Mathematical Surveys, **32:3**, 1–57 (1977).
- [4] H. Busemann, *The Geometry of Geodesics*, Academic Press, New-York, 1955.
- [5] H. Busemann and B. Phadke, *Peakless and monotone functions on G-spaces*, Tsukuba J. Math. **7**, 105–135 (1983).
- [6] J. Cheeger–D. Ebin, *Comparison Theorems in Riemannian Geometry*, AMS Chelsea Publ. Amer. Math. Soc. Providence, Rhode Island, 2008.
- [7] R. Greene–K. Shiohama, *Convex functions on complete noncompact manifolds; Topological structure*, Invent. Math. **63**, 129–157 (1981).

- [8] R. Greene–K. Shiohama, *Convex functions on complete noncompact manifolds: Differentiable structure*, Ann. Scient. Éc. Norm. Sup. **14**, 357–367 (1981).
- [9] R. Greene–K. Shiohama, *The isometry groups of manifolds admitting nonconstant convex functions*, J. Math. Soc. Japan, **39** 1–16 (1987).
- [10] N. Innami, *Splitting theorems of Riemannian manifolds*, Comp. Math. **47** 237–247 (1982).
- [11] N. Innami, *A classification of Busemann  $G$ -surfaces which possess convex functions*, Acta Math. **148**, 15–29 (1982).
- [12] N. Innami, *Totally flat foliations and peakless functions*, Archiv Math. **41**, 464–471 (1983).
- [13] Y. Mashiko, *Convex functions on Alexandrov surfaces*, Trans. Amer. Math. Soc. **351**, no. 9, 3549–3567 (2006).
- [14] Y. Mashiko, *Affine functions on Alexandrov surfaces*, Osaka J. Math. **36**, 853–859 (1999).
- [15] Y. Mashiko, *A splitting theorem for Alexandrov spaces*, Pacific J. Math. **204**, 445–458 (2002).
- [16] S. S. Chern–W. H. Chen–K. S. Lam, *Lectures on differential Geometry*, World Scientific Publ. Co. 2000.
- [17] S. V. Sabau, *On the convexity of Finslerian metric balls, convexity radius and related topics*, Preprint, 2013.
- [18] T. Sakai, *Riemannian Geometry*, Mathematical Monograph, Amer. Math. Soc. **8** (1997).
- [19] V. Sharafutdinov, *Pogorelov-Klingenberg theorem for manifolds homeomorphic to  $\mathbb{R}^n$* , Siberian Math. J. **18**, 915–925 (1977).
- [20] J. H. C. Whitehead, *On the covering of a complete space by the geodesics through a point*, Ann. of Math. (2) **36**, (1935), No.3, 679–704.
- [21] T. Yamaguchi, *The isometry groups Riemannian manifolds admitting strictly convex functions*, Ann. Scient. École Norm. Sup. **15** 205–212 (1982).
- [22] T. Yamaguchi, *Locally geodesically convex functions on complete Riemannian manifolds*, Trans. Amer. Math. Soc., **298**, 307–330 (1986).
- [23] T. Yamaguchi, *Uniformly locally convex filtrations on complete Riemannian manifolds*, Curvature and topology of Riemannian manifolds (Katata, 1986), 308–316, Lecture Notes in Math., **s1203**, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo.

- [24] T. Yamaguchi, *On the structure of locally convex filtration on complete manifolds*, J. Math. Soc. Japan, **40**, 221–234, (1980).